Journal of Algebra **243**, 765–779 (2001)
doi:10.1006/jabr.2001.8851, available online at http://www.idealibrary.com on IDEAL®

Modules whose Injective Endomorphisms Are Essential

A. Haghany

Institute for Studies in Theoretical Physics and Mathematics, Tehran, Iran; and

and

M. R. Vedadi

Department of Mathematics, Isfahan University of Technology, Isfahan, Iran

Communicated by Gernot Stroth

Received January 12, 2001

An R-module M is called weakly co-Hopfian if any injective endomorphism of M is essential. The class of weakly co-Hopfian modules lies properly between the class of co-Hopfian and the class of Dedekind finite modules. Several equivalent conditions are given for a module to be weakly co-Hopfian. Being co-Hopfian, weakly co-Hopfian, or Dedekind finite are all equivalent conditions on quasi-injective modules. Some other properties of weakly co-Hopfian modules are also obtained. The ring R is said to be right strong stably finite if all the finitely generated free right R-modules are weakly co-Hopfian. We shall characterize such rings and show that they are stably finite and satisfy the right strong rank condition. Examples show that stably finite rings and rings with the right strong rank conditions need not be strong stably finite. Both weakly co-Hopfian and right strong stably finite are Morita invariants, although the right and left strong stably finite are different properties. The class of commutative rings and the class of rings with finite right uniform dimension are proper subclasses of the class of right strong stably finite rings. We shall also investigate conditions that are relevant to weakly co-Hopfian modules. Equivalent statements are found on a ring to have all its finitely generated right modules weakly co-Hopfian. © 2001 Academic Press

1. WEAKLY CO-HOPFIAN MODULES

Rings will have unit elements and modules will be unitary. The terminology not defined here may be found in [6]. Let R be a ring and M_R a right R-module. M is called co-Hopfian if any injective endomorphism of M is

0021-8693/01 $35.00

an isomorphism; see [8, 9] for a discussion of such modules. We say an R-module M is *weakly co-Hopfian* if any injective endomorphism f of M is essential; that is, $f(M) \subseteq_e M$.

THEOREM 1.1. *The following are equivalent conditions on a right R-module M.*

(1) *M_R is weakly co-Hopfian.*

(2) *For any right R-module N, if there is an R-monomorphism $M \oplus N \to M$, then $N = 0$.*

(2′) *For any right R-module N, if $M \oplus N \to M$ is an essential monomorphism then $N = 0$.*

(3) *M is Dedekind finite and the image of any injective endomorphism of M is either essential or a proper direct summand.*

(4) *There exists a fully invariant essential submodule which is weakly co-Hopfian.*

(5) *Injective endomorphisms of M_R map essential submodules to essential submodules.*

(6) *The inverse image of any nonzero submodule under any injective endomorphism of M is nonzero.*

Proof. We shall prove that (1) ⇒ (2) ⇒ (2′) ⇒ (1), (2) ⇒ (3) ⇒ (1), and (4) ⇒ (1) ⇔ (6); (1) ⇒ (4) and (1) ⇔ (5) are trivial.

(1) ⇒ (2). Suppose f: $M \oplus N \to M$ is a monomorphism and N is a right R-module. If ι: $M \to M \oplus N$ is the canonical injection then f_ι is an injective endomorphism of M, hence f_ι has an essential image. If now $N \neq 0$, then $f_\iota(M) = f(M \oplus 0)$ intersects the nonzero submodule $f(0 \oplus N)$ nontrivially, and this is impossible.

(2) ⇒ (2′). This is trivial.

(2′) ⇒ (1). Let g be an injective endomorphism of M with nonessential image. Then there exists a nonzero submodule K with $g(M) \oplus K \subseteq_e M$. But this yields an essential monomorphism $M \oplus K \to M$, contradicting (2′). Hence $g(M) \subseteq_e M$.

(2) ⇒ (3). From a well-known characterization of Dedekind finite modules we have M Dedekind finite. Since (2) is equivalent to (1), the image of any injective endomorphism of M is in fact an essential submodule.

(3) ⇒ (1). Let g: $M \to M$ be injective and $g(M)$ not be essential in M. Then by assumption $g(M) \oplus K = M$ for some nonzero submodule K. Hence there is an isomorphism $M \oplus K \simeq M$ which contradicts the Dedekind finiteness of M. Thus $g(M) \subseteq_e M$.

(4) $\Rightarrow$ (1). Suppose $K \subseteq_e M$, with K fully invariant and weakly co-Hopfian. Let g be an injective endomorphism of M. Then $g|_K$ is an injective endomorphism of K, hence $g(K) \subseteq_e K$. Since $K \subseteq_e M$, we deduce that $g(K) \subseteq_e M$. But then $g(M) \subseteq_e M$.

(1) $\Rightarrow$ (6). Let N be a nonzero submodule and f an injective endomorphism of M. Then $f(M) \cap N \neq 0$, so if n is a nonzero element of N with $n = f(m)$ for some $m \in M$ then we have $0 \neq m \in f^{-1}(f(M) \cap N) = f^{-1}(f(M)) \cap f^{-1}(N) = M \cap f^{-1}(N) = f^{-1}(N)$.

(6) $\Rightarrow$ (1). If there is an injective endomorphism g of M such that $g(M) \cap N = 0$ for some nonzero submodule N then

$$g^{-1}(g(M) \cap N) = g^{-1}(0) = 0,$$

which implies $g^{-1}(N) = 0$, contradicting (6).

COROLLARY 1.2. *If M_R is weakly co-Hopfian and f is an injective endomorphism of M, then*:

(1) *$N \subseteq_e M$ if and only if $f(N) \subseteq_e M$ if and only if $f^{-1}(N) \subseteq_e M$.*

(2) Soc $M = \cap f(N) = \cap f^{-1}(N)$, *where N runs through the set of all essential submodules of M.*

COROLLARY 1.3. *The following hold.*

(i) *A direct summand of a weakly co-Hopfian module is weakly co-Hopfian.*

(ii) *A module is weakly co-Hopfian whenever its injective envelope is Dedekind finite.*

PROPOSITION 1.4. *For an R-module M, consider the following statements.*

(1) *M is co-Hopfian.*

(2) *M is weakly co-Hopfian.*

(3) *M is Dedekind finite.*

Then (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3). *If M is quasi-injective all three statements are equivalent.*

Proof. Clearly every co-Hopfian module is weakly co-Hopfian, and the implication (2) $\Rightarrow$ (3) is evident from Theorem 1.1. Next assume that M is quasi-injective and Dedekind finite. Let h be an injective endomorphism of M. Then $h(M) \simeq M$, and so $h(M)$ is M-injective. Thus $h(M)$ is a direct summand of M; i.e., there exists a submodule K in M such that $h(M) \oplus K = M$. Hence $M \oplus K \simeq M$, which by (3) implies $K = 0$. Therefore $h(M) = M$ and M is co-Hopfian, showing that (3) $\Rightarrow$ (1).

COROLLARY 1.5. *Let M be quasi-injective.*

(i) *If M is weakly co-Hopfian so are any submodule and any finite direct sum of copies of M.*

(ii) *Suppose N is a fully invariant essential submodule of M. Then N is weakly co-Hopfian if and only if so is M. Moreover, M is weakly co-Hopfian if and only if so is $E(M)$.*

Proof. (i) The weak co-Hopficity of M^n is immediate from Proposition 1.4 as M^n is Dedekind finite by [1, Corollary 6.21] and quasi-injective by [6, Corollary 6.77]. Now to prove that any submodule is weakly co-Hopfian it suffices, by Corollary 1.3, to show this for any essential submodule N of M. So assume that f: $N \to N$ is a monomorphism. Since M is quasi-injective there exists $g \in \operatorname{End}(M)$ such that $g|_N = f$. Clearly $N \cap \ker g = 0$, so $\ker g = 0$ and g is a monomorphism. By Theorem 1.1, $g(N) \subseteq_e M$, and so $f(N) \subseteq_e N$. Thus N is weakly co-Hopfian.

(ii) The first assertion is obvious from Theorem 1.1 and (i) above. For the second, we just recall that M is a fully invariant essential submodule of $E(M)$ and then apply the first assertion.

LEMMA 1.6. *A cyclic right R-module R/I is weakly co-Hopfian if and only if for any element $a \in R$ with a: $I = I$ and for any right ideal J the following is valid*:

$$(aR + I) \cap (J + I) = I \Rightarrow J \subseteq I.$$

Proof. Any R-endomorphism f of R/I is given by left multiplication by some element $a \in R$ such that $aI \subseteq I$. Further, f is a monomorphism if and only if $a : I = I$. Clearly f is essential if and only if $aR + I/I$ is essential in R/I and this is equivalent to the condition stated in Lemma 1.6.

Remarks 1.7. (i) Specializing I to be the zero ideal, we see that R_R is weakly co-Hopfian if and only if any right regular element generates an essential right ideal in R. Consequently a commutative ring is always weakly co-Hopfian as a module over itself.

(ii) Note that a commutative ring that is not its own total quotient ring provides us with a non-co-Hopfian module which is weakly co-Hopfian.

The following examples show that the class of weakly co-Hopfian modules lies properly in between classes of co-Hopfian and Dedekind finite modules and that modules with finite uniform dimension form a proper subclass of weakly co-Hopfian modules.

EXAMPLES 1.8. (1) Any module with finite uniform dimension is weakly co-Hopfian. However, if $R = \Pi Z_p$ where p runs through the set of all prime numbers then R_R has infinite uniform dimension, yet it is weakly co-Hopfian in Mod-R.

(2) Let F be a field, having a monomorphism $\sigma: F \to F$ that is not onto, and let $R = F[x; \sigma]$ be the ring of polynomials $a_0 + xa_1 + \cdots + x^n$, where $a_i \in F$, subject to $ax = x\sigma(a)$. It is known that R is an integral domain with right uniform dimension 1 and infinite left uniform dimension. Thus all finitely generated free right R-modules are weakly co-Hopfian. However, the left regular module ${}_RR$ is Dedekind finite but not weakly co-Hopfian in R-Mod. In fact, right multiplication by x is an injective endomorphism of ${}_RR$ whose image Rx is not an essential left ideal. It has zero intersection with Rxa, where a is any element of F not in $\sigma(F)$.

PROPOSITION 1.9. *Let M_R be a right R-module with any proper submodule weakly co-Hopfian. Then M is weakly co-Hopfian.*

Proof. Let f: $M \to M$ be a monomorphism with $N = f(M)$. If N is proper then by assumption N is weakly co-Hopfian, hence so is $M \simeq f(M)$. On the other hand, if any such f is onto, then M is co-Hopfian and hence weakly co-Hopfian.

Remarks 1.10. (i) In view of Corollary 1.5(ii) and Proposition 1.9, we have that if M is quasi-injective then M is weakly co-Hopfian if and only if so is any proper submodule of M.

(ii) Let M be a nonzero module, $M' = \oplus_{i \geq 1} M_i$ where $M_i = M$ for any $i \geq 1$. The "shift" map on M' is an injective endomorphism whose image is not an essential submodule. Thus M' is never weakly co-Hopfian. This observation together with Corollary 1.5 yields the fact that if M is a simple module, then a direct sum of the copies of M is weakly co-Hopfian if and only if the sum is finite. A more general result will be stated below in Corollary 1.12.

(iii) Submodules of weakly co-Hopfian modules need not be weakly co-Hopfian: Let G be an infinite direct sum of Z_2 and $R = G \oplus Z$ the ring with identity and zero ring structure on G. Then $G \oplus 0$ is an ideal in the commutative ring R such that its R-endomorphisms coincide with the endomorphisms of the abelian group G. Thus $G \oplus 0$ fails to be weakly co-Hopfian in Mod-R.

PROPOSITION 1.11. *Let $M = \sum_{i \in I} \oplus M_i$ such that each M_i is invariant under any injective endomorphism of M. Then M is weakly co-Hopfian if and only if so is each M_i for $i \in I$.*

Proof. If f is an injective endomorphism of M, then by the weak co-Hopficity of M_i, we get $f(M_i) \subseteq_e M_i$, and consequently $\Sigma_i \oplus f(M_i) \subseteq_e \Sigma_i \oplus M_i$, hence $f(M) \subseteq_e M$.

COROLLARY 1.12. *A semisimple module M is weakly co-Hopfian if and only if any homogeneous component of M is finitely generated.*

Proof. This is immediate from Proposition 1.11 and Remarks 1.10(ii).

PROPOSITION 1.13. *Let N be a fully invariant submodule of M_R. Suppose that N is co-Hopfian. If M/N is co-Hopfian (resp. weakly co-Hopfian) then so is M.*

Proof. Let f: $M \to M$ be an R-monomorphism, $g = f|_N$. Then $g(N) = N$ from our assumptions. Thus $f(N) = N$, so the rule $\bar{f}(m + N) = f(m) + N$ defines a well-defined monomorphism $\bar{f}$: $M/N \to M/N$. If M/N is weakly co-Hopfian, $f(M)/N \subseteq_e M/N$. Thus $f(M) \subseteq_e M$, and M is weakly co-Hopfian. If M/N is co-Hopfian then $\bar{f}$ is onto, giving f onto, and M is co-Hopfian.

PROPOSITION 1.14. *If d.c.c. holds on nonessential submodules of M_R then M is weakly co-Hopfian.*

Proof. Suppose M is not weakly co-Hopfian. Then there exists $M_1 \subseteq M$ such that $M_1 \simeq M$ and $M_1 \not\subseteq_e M$. Now M_1 is not weakly co-Hopfian, so there exists $M_2 \subseteq M_1$ such that $M_2 \simeq M_1$ and $M_2 \not\subseteq_e M_1$. But then $M_2 \not\subseteq_e M$ and $M_2 \simeq M$. So we can repeat to produce a strict descending chain of nonessential submodules.

Remark 1.15. The converse of Proposition 1.14 is not true in general: $\Sigma_p \oplus Z_p$ is co-Hopfian (so weakly co-Hopfian) in Mod-Z but does not satisfy d.c.c. on nonessential submodules.

Our next result deals with co-Hopfian modules.

PROPOSITION 1.16. *Let $\mathscr{P}$ be a property of modules preserved under isomorphism. If a module M has the property $\mathscr{P}$ and satisfies d.c.c. on submodules with property $\mathscr{P}$ then M is co-Hopfian.*

Proof. Suppose M is not co-Hopfian. Then there exists a proper submodule N_1 of M with $N_1 \simeq M$. Thus N_1 is not co-Hopfian and enjoys $\mathscr{P}$. We have a proper submodule N_2 of N_1 with $N_2 \simeq N_1$. Clearly N_2 is not co-Hopfian and satisfies $\mathscr{P}$. Repeating we obtain a strictly descending chain $N_1 \supsetneqq N_2 \supsetneqq \cdots$ of proper submodules each with property $\mathscr{P}$, a contradiction.

COROLLARY 1.17. *If M is Hopfian and has d.c.c. on Hopfian submodules then M is co-Hopfian.*

COROLLARY 1.18. *If M has d.c.c. on its non-co-Hopfian submodules then M is co-Hopfian.*

Proof. Suppose not, and let $\mathscr{P}$ be the property of being non-co-Hopfian. Applying Proposition 1.16 we arrive at a contradiction. Thus M must be co-Hopfian.

The last result in this section concerns a kind of relative (weak) co-Hopficity. If A and B are right R-modules we say B is weakly co-Hopfian (resp. co-Hopfian) related to A if for any monomorphism f: $A \to B$ we have $f(A) \subseteq_e B$ (resp. $f(A) = B$). If A and B are isomorphic co-Hopfian modules then each module is co-Hopfian related to the other. If A and B are atomic modules with the same Noetherian dimension, then they are co-Hopfian as well as co-Hopfian related, but not necessarily isomorphic; see [5, Proposition 2.2(3) and Proposition 3.3]. For any prime number p we have that Z is weakly co-Hopfian related to the localized module $Z_{(p)}$ in Mod-Z but not co-Hopfian related.

PROPOSITION 1.19. *If any two nontrivial factor modules of M are (weakly) co-Hopfian related to each other then M is (weakly) co-Hopfian.*

Proof. A factor, M/N, is nontrivial when $0 \neq N \subsetneqq M$. Let f be an injective endomorphism of M and suppose that there exists a nonzero proper submodule, say N, of $f(M)$. If no such N exists M is simple and the conclusions are clear. Now $0 \neq f^{-1}(N) \neq M$, and f induces a monomorphism $\bar{f}$: $M/f^{-1}(N) \to M/N$ given by $\bar{f}(x + f^{-1}(N)) = f(x) + N$. From our assumption it follows that $\bar{f}$ is essential or onto. But Im $\bar{f}$ $= \frac{f(M)+N}{N} = \frac{f(M)}{N}$, so the required conclusions follow.

2. STRONG STABLY FINITE RINGS

Recall from [6] that a ring R is called stably finite if all the matrix rings $\mathrm{Mat}_{n\times n}(R)$ are Dedekind finite and that this is equivalent to the Hopficity of all finitely generated free (right) R-modules. Motivated by this, we consider rings all of whose finitely generated free right modules are weakly co-Hopfian. Immediate examples of such rings are rings of finite right uniform dimension. As a result of Corollary 1.3, over such rings all finitely generated projective right modules are also weakly co-Hopfian.

DEFINITION 2.1. A ring R is called right strong stably finite if R_R^n is weakly co-Hopfian for all integers $n \geq 1$.

PROPOSITION 2.2. *If R is right strong stably finite then R is stably finite. The converse holds if R_R is injective.*

Proof. Suppose R is right strong stably finite and let n be a positive integer such that $R^n \oplus K \simeq R^n$ for some $K \in$ Mod-R. Clearly $R^n \oplus K$ embeds in R^n from which $K = 0$ follows by Theorem 1.1. Thus by [6, 1.7] R is stably finite. Assume that R_R is injective and stably finite. By Proposition 1.4, R_R is weakly co-Hopfian, hence so is R_R^n for any n by Corollary 1.5(i). Therefore R is right strong stably finite.

Remark 2.3. The ring of Examples 1.8(2) shows that strong stably finite is not a symmetric condition.

Rings with the right strong rank condition (see [6]) form another proper subclass of right strong stably finite rings.

PROPOSITION 2.4. (i) *Let R be right strong stably finite. Then R has the right strong rank condition.*

(ii) *If R is an integral domain with right strong rank condition then R is a right strong stably finite ring.*

Proof. (i) Assume $0 \to R^m \to R^n$ is exact in Mod-R with $m > n$. Then $R^n \oplus R^{m-n} \simeq R^m$ which embeds in R^n. Since R^n is assumed weakly co-Hopfian we arrive at the contradiction $R^{m-n} = 0$. Therefore R has the right strong rank condition.

(ii) This is clear from the fact [6, 1.32] that for an integral domain the right strong rank condition is equivalent to being right Ore, hence being uniform. There is an easy alternative proof as follows: Suppose R is an integral domain with right strong rank condition and $N \subseteq R^k$ such that $N \simeq R_R^k$. If N is not an essential submodule, there exists a nonzero submodule L in R^k with $N \cap L = 0$. For any nonzero $l \in L$, $lR \simeq R$ since R is an integral domain, and so

$$R^k \oplus R \simeq N \oplus lR \subseteq R^k.$$

This contradicts that R has the right strong rank condition. Hence N is essential and it follows that R is right strong stably finite.

Remarks 2.5. (a) We shall strengthen 2.4(ii) in Proposition 3.1 below.

(b) Take a stably finite ring A that does not have the right strong rank condition (see [6, 1.34]). Then by Proposition 2.4, A is not right strong stably finite.

(c) Let B be a ring with the right strong rank condition and C a ring that is not stably finite. Then $B \times C$ has the right strong rank condition but it is not stably finite, and consequently by Proposition 2.2 $B \times C$ is not right strong stably finite.

Next we show that right strong stably finite is a Morita invariant property.

THEOREM 2.6. *Weakly co-Hopfian and right strong stably finite are Morita invariant properties.*

Proof. Let R and S be Moritz equivalent rings with inverse category equivalences α: Mod-$R \to$ Mod-S and β: Mod-$S \to$ Mod-R. Suppose $M \in$ Mod-R is weakly co-Hopfian. To show that $\alpha(M)$ is a weakly co-Hopfian object in Mod-S, let f: $\alpha(M) \oplus K \to \alpha(M)$ be a monomorphism in Mod-S and K a right S-module. Since any category equivalence preserves monomorphisms and direct sums, we obtain $\beta(f)$: $\beta\alpha(M) \oplus \beta(K) \to \beta\alpha(M)$, a monomorphism in Mod-$R$. Hence we have a monomorphism $M \oplus \beta(K) \to M$ which by weak co-Hopficity of M implies $\beta(K) = 0$. Consequently $K = 0$ and so weak co-Hopfian is a categorical property.

We now assume R is right strong stably finite. Consider a finitely generated free right S-module S^n. Then $\beta(S^n)$, being a progenerator, is a direct summand of some R^m_M. By assumption R^m_R is weakly co-Hopfian, hence so is $\beta(S^n)$ by Corollary 1.3. By the first part of the proof $\alpha\beta(S^n) \simeq S^n_S$ is weakly co-Hopfian. Therefore S is right strong stably finite.

We now give a characterization of right strong stably finite rings and then use it in proving that all commutative rings are strong stably finite. This characterization is a natural generalization of weak co-Hopficity for R_R.

PROPOSITION 2.7. *A ring R is right strong stably finite if and only if, for any $n \geq 1$, if $u_1, \dots, u_n$ are R-linearly independent vectors in R^n_R then $u_1R + \cdots + u_nR$ is an essential submodule of R^n_R.*

Proof. Suppose R is right strong stably finite and let $u_1, \dots, u_n$ be R-linearly independent vectors in R^n. The map f: $R^n \to R^n$ given by $f(r_1, \dots, r_n) = u_1r_1 + \cdots + u_nr_n$ is a right R-monomorphism with $\operatorname{Im} f = u_1R + \cdots + u_nR$. By weak co-Hopficity of R^n we have $\operatorname{Im} f \subseteq_e R^n_R$. Conversely, suppose g is an injective endomorphism of R^n_R. Set $g(e_i) = u_i$ where $e_i = (0, \dots, 1, 0, \dots, 0)$. Since g is injective $u_1r_1 + \cdots + u_nr_n = 0$ will imply $r_1 = \cdots = r_n = 0$. Thus $u_1, \dots, u_n$ are R-linearly independent vectors, and so $\operatorname{Im} g = u_1R + \cdots + u_nR$ is an essential submodule by assumption.

THEOREM 2.8. *Every commutative ring is strong stably finite.*

Proof. We apply Proposition 2.7 to the commutative ring R. Thus let $u_1, \dots, u_n$ be R-linearly independent vectors in R^n and set $I = u_1R + \cdots + u_nR$. We show that for any nonzero vector $u_{n+1} \in R^n$, $I \cap u_{n+1}R \neq 0$. Write $u_j = \sum_i e_i a_{ij}$ ($j = 1, \dots, n+1$ and $i = 1, \dots, n$) where $a_{ij} \in R$ and denote by S the subring of R generated by elements a_{ij} and 1_R. By Hilbert Basis Theorem S is a Noetherian ring, hence S_S has finite uniform dimension. Consequently S is a strong stably finite ring. Clearly $u_1, \ldots, u_{n+1} \in S^n$ and $u_1, \ldots, u_n$ are S-linearly independent. Hence $(u_1 S + \cdots + u_n S) \cap u_{n+1} S \neq 0$ by Proposition 2.7 and so $I \cap u_{n+1} R \neq 0$. It follows that $I \subseteq_e R^n$.

COROLLARY 2.9. *If T is a ring Morita equivalent to a commutative ring then T is right and left strong stably finite.*

Proof. Just apply Theorem 2.8 and Theorem 2.6.

3. FURTHER RESULTS RELATED TO WEAKLY CO-HOPFIAN MODULES

Recall that R is co-Hopfian in Mod-R (resp. R-Mod) if every right (res. left) regular element is a unit. Thus,

> R is a semisimple ring if and only if R_R (resp. ${}_R R$) is co-Hopfian and every essential right (resp. left) ideal contains a right (res. left) regular element.
>
> Since R is weakly co-Hopfian in Mod-R if and only if every right regular element generates an essential right ideal, a characterization of semiprime right Goldie rings (see [6, 11.13]) can be written as
>
> A ring R is semiprime right Goldie if and only if R_R is weakly co-Hopfian and every essential right ideal contains a regular element.

PROPOSITION 3.1. *Let R be an integral domain. The following are equivalent.*

(1) *R_R is weakly co-Hopfian.*

(2) *R is right strong stably finite.*

(3) *R satisfies the right strong rank condition.*

(4) *Any two elements in R are right R-linearly dependent.*

Proof. If R_R is weakly co-Hopfian, then by the preceding observation the domain R satisfies the right Goldie conditions, hence it is a right Ore domain. But then R is right uniform and (2) follows. Since by [6, 1.32] for integral domains the right Ore condition is equivalent to the right strong rank condition, we have (1) iff (3). Clearly (3) $\Rightarrow$ (4) and (2) $\Rightarrow$ (1). Now assume (4), and let c be a nonzero element in R. For any nonzero $d \in R$, by (4) we have $x, y \in R$ not both zero with $cx + dy = 0$. But in fact $x = 0$ iff $y = 0$. This means $cR \cap dR \neq 0$, hence $cR \subseteq_e R_R$. Thus (4) $\Rightarrow$ (1).

PROPOSITION 3.2. *Let R be a ring and consider the following conditions.*

(1) *Any right ideal of R is weakly co-Hopfian.*

(2) *All essential right ideals of R are weakly co-Hopfian.*

(3) *If I is a right ideal and $u \in R$ such that $r.\mathrm{ann}_R u \cap I = 0$ and $uI \subseteq I$ then $uI \subseteq_e I$.*

(4) *If I is an essential right ideal and u is a right regular element of R with $uI \subseteq I$ then $uI \subseteq_e R$.*

Then (1) $\Leftrightarrow$ (2) and (2) $\Rightarrow$ (3) $\Rightarrow$ (4). Furthermore, if R is right self-injective then all four statements are equivalent.

Proof. We only prove that if R_R is injective then (4) $\Rightarrow$ (2). So suppose f is an injective R-endomorphism of an essential right ideal I. There exists an R-endomorphism g of R_R with $g|_I = f$. Hence $\ker g \cap I = \ker f = 0$, so $\ker g = 0$ and g is a monomorphism. But then there exists a right regular element $u \in R$ with $g(r) = ur$ for any $r \in R$. So we have $uI = g(I) = f(I) \subseteq I$, hence, by (4), $uI \subseteq_e R_R$. This implies that $uI \subseteq_e I$.

In order to state our next result in a convenient way we introduce some terminology. Considering elements of the free right R-module R^n as column matrices we say a matrix $A \in \mathrm{Mat}_{n\times n}(R)$ acts regularly on a submodule K of R_R^n iff for any $x \in R^n$,

$$Ax \in K \Leftrightarrow x \in K.$$

Moreover, A is said to act essentially mod K iff the following condition is valid:

$$\text{For any } y \in R^n - K \ \exists r \in R \text{ and } x \in R^n \text{ such that } yr \notin K \text{ and } Ax - yr \in K.$$

THEOREM 3.3. *The following are equivalent statements for a ring R.*

(1) *All finitely generated right R-modules are weakly co-Hopfian.*

(2) *For any n, if $A \in \mathrm{Mat}_{n\times n}(R)$ acts regularly on a right R-submodule K of R^n then A acts essentially mod K.*

(3) *For any n, if $A \in S = \mathrm{Mat}_{n\times n}(R)$, with I and J right ideals in S, the following implication holds*:

$$A : I = I \quad \text{and} \quad (AS + I) \cap (J + I) = I \Rightarrow J \subseteq I.$$

Proof. Let n be a positive integer. Using the standard Morita equivalence of Mod-R with Mod-S where $S = \mathrm{Mat}_{n\times n}(R)$ we know that a right R-module generated by n elements correspond to a cyclic right S-module, and conversely a cyclic right S-module corresponds to a finitely generated right R-module. Thus the equivalence of (1) with (3) is clear in view of Theorem 2.6 and Lemma 1.6.

Suppose now that M is a finitely generated right R-module. We can assume that $M \simeq R^t/K$ for some integer t and right R-submodule K of R^t. If f is an R-endomorphism of R^t/K, we let $f(e_i + K) = u_i + K$ for $i = 1, \dots, t$. Form the $t \times t$ matrix A whose ith column is u_i. Then f is a monomorphism if and only if A acts regularly on K. Moreover, $\operatorname{Im} f = \Sigma u_i R + K/K$ is essential in R^t/K precisely when A acts essentially mod K. From these observations we deduce that (2) $\Rightarrow$ (1).

Conversely, given a matrix $A \in \operatorname{Mat}_{n \times n}(R)$ acting regularly on a right R-submodule K of R^n we see that the rule

$$f(x + K) = Ax + K$$

defines a well-defined monomorphism f: $R^n/K \to R^n/K$. By (1), f is essential, hence A acts essentially mod K. This shows that (1) $\Rightarrow$ (2).

In the following the condition stated in (i) is weaker than the condition that guarantees (in fact is equivalent to) a ring will be semiprime right Goldie.

THEOREM 3.4. *The following statements are equivalent on a ring R.*

(i) *For all two-sided ideals I of R, $I \subseteq_e R_R$ if and only if I contains regular elements.*

(ii) *Any prime ideal which is essential as a right ideal contains regular elements (the right π-condition).*

Proof. Trivially (i) implies (ii). Assume (ii) holds. We first show that if I is a two-sided ideal and $I \subseteq_e R_R$ then I contains regular elements. Suppose not, and choose by Zorn's lemma an ideal $J \supseteq I$ maximal with respect to not meeting the multiplicatively closed set of regular elements of R. But then J is a prime ideal and $J \subseteq_e R_R$, which contradicts (ii). Next suppose I contains a regular element c, say. Choose a right ideal K with $I \oplus K \subseteq_e R_R$. Since $KI \subseteq I \cap K = 0$ we have $Kc = 0$ and consequently $K = 0$. This yields $I \subseteq_e R_R$, and so (i) follows.

COROLLARY 3.5. *For a right bounded ring the right π-condition is equivalent to*

($r * 2$ *condition*) *Every essential right ideal contains a regular element.*

COROLLARY 3.6. *The following are equivalent conditions on a commutative ring R.*

(i) *R satisfies the $r * 2$ condition.*

(ii) *R is semiprime right Goldie.*

(iii) *R satisfies the right π-condition.*

Let R be semiprime right Goldie. According to [2, Theorem 6.15] $R \simeq S \times T$ where S is a semisimple ring and $\operatorname{Soc} T_T = 0$. Nicholson and Yousif have proved that for an I-finite ring the existence of such a ring decomposition is equivalent to the right universal mininjectivity, see [7, Theorem 5.2]. We say that R satisfies the $r * r$ (resp. $l * l$) condition if any essential right (resp. left) ideal contains a right (resp. left) regular element. Such a ring is necessarily semiprime. Now only a minor variation in the proof of [2, Theorem 6.15] using Theorem 1.1 yields the following structural result.

PROPOSITION 3.7. *Let R be a ring with $r * r$ condition. If $\operatorname{Soc} R_R$ is weakly co-Hopfian then* $\operatorname{Soc} R_R$ *is generated by a central idempotent $e \in R$ and consequently there is a ring decomposition $R \simeq S \times T$ where $S = \operatorname{Soc} R_R = eR$ is a semisimple ring and $T = (1 - e)R$ has zero right socle.*

Using Proposition 3.7 and its left hand analogue we have

COROLLARY 3.8. *The following are equivalent on a ring R.*

(1) *R is simple Artinian.*

(2) *R is a prime ring with $r * r$ condition and nonzero (weakly) co-Hopfian right socle.*

(3) *R is a prime ring with $l * l$ condition and nonzero (weakly) co-Hopfian left socle.*

Remark 3.9. There are rings with infinite uniform dimension satisfying the hypotheses of Proposition 3.7. For example let F be a field, T an integral domain with infinite right uniform dimension, and $R = F \times T$. Clearly R_R has infinite right uniform dimension and since a finite direct product of integral domains satisfies $r * 2$, the ring R has $r * r$ condition. Also, $\operatorname{Soc} T_T = 0$ because an integral domain with nonzero socle is a division ring. Consequently $\operatorname{Soc} R_R = F \times 0$ which is co-Hopfian in Mod-R.

It is known that a prime ideal is either a minimal prime ideal or essential as a right ideal. It is then not out of place to consider the following condition.

(r.minπ-condition) No minimal prime ideal of R is essential as a right ideal.

EXAMPLES 3.10. 1. If R is a semiprime right Goldie ring, then by [2, Proposition 6.3] R satisfies the r.min π-condition.

2. Let A be the right of integers, B the field of two elements, and

$$R = \begin{bmatrix} A & 0 \\ B & A \end{bmatrix}.$$

Then R is not semiprime, and it has precisely two minimal prime ideals, namely

$$\begin{bmatrix} A & 0 \\ B & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 0 \\ B & A \end{bmatrix}.$$

None of these is essential as a right ideal. This can be seen either directly or by [3, Proposition 1.1]. Thus R satisfies the $r.\min\,\pi$-condition.

We collect in our final result some consequences of the $r.\min\,\pi$-condition. The proof of Part (c) in Proposition 3.11, based on [4, 1.1.5 and 1.1.6], is included for the reader's convenience.

PROPOSITION 3.11. *Let R be a ring.*

(a) *If R satisfies the $r.\min\,\pi$-condition and R_R is uniform then R is a prime ring.*

(b) *R is a division ring if and only if R_R is uniform and R has the $r.\min\,\pi$-condition and nonzero right socle.*

(c) *R is a commutative domain if and only if in R there exists a nonzero commutative ideal, R_R is uniform, and the $r.\min\,\pi$-condition holds.*

Proof. (a) Any ring has minimal prime ideals. Let P be a minimal prime ideal in R. If P is nonzero then $P \subseteq_e R_R$, since R_R is uniform. But this contradicts the $r.\min\,\pi$-condition. Hence $P = 0$ and R is a prime ring.

(b) Suppose R_R is uniform, $\mathrm{Soc}(R_R) \neq 0$, and the $r.\min\,\pi$-condition holds in R. Let I be a minimal right ideal. Since R is a prime ring, I is generated by an idempotent element, and so I is a direct summand of R. Since R_R is uniform, $I = R$, hence R is a division ring. The converse is clear.

(c) One implication is obvious. So now let I be a nonzero ideal in R such that for all $x, y \in I$, $xy = yx$, and R is right uniform and satisfies the $r.\min\,\pi$-condition. Let $x, y \in I$ and $r \in R$. Since $ry \in I$, we have

$$(xr - rx)y = x(ry) - rxy = ryx - ryx = 0.$$

Thus $xr - rx \in l.\mathrm{ann}_R I$ which is zero as R is a prime ring by (a). It follows that I is inside the center of R. Hence if $a, b \in R$ and $x \in I$, since $bx \in I$ we obtain $(ab - ba)x = 0$. Therefore $ab = ba$ as before.

REFERENCES

1. K. R. Goodearl, "Ring Theory: Nonsingular Rings and Modules," Dekker, New York, 1976.

2. K. R. Goodearl and R. B. Warfield, Jr., "An Introduction to Non-commutative Noetherian Rings," London Mathematical Society Student Texts, Vol. 16, London Math Society, London, 1989.
3. A. Haghany and K. Varadarajan, Study of modules over formal triangular matrix rings, *J. Pure Appl. Algebra* **147** (2000), 41–58.
4. I. N. Herstein, "Rings with Involution," University of Chicago Press, Chicago, 1976.
5. O. A. S. Karamzadeh and A. R. Sajedi Nejad, Atomic modules, *Comm. Algebra*, to appear.
6. T. Y. Lam, "Lectures on Modules and Rings," Graduate Texts in Mathematics, Vol. 139, Springer-Verlag, New York/Berlin, 1998.
7. W. K. Nicholson and M. F. Yousif, Mininjective rings, *J. Algebra* **187** (1997), 548–578.
8. K. Varadarajan, Hopfian and co-Hopfian objects, *Publ. Mat.* **36** (1992), 293–317.
9. K. Varadarajan, Properties of endomorphism rings, *Acta Math. Hungar.* **74** (1997), 83–92.